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Ferromagnetic pairing states on two-coupled chains

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Abstract

We propose a concrete model which exhibits ferromagnetism and electron-pair condensation simultaneously. The model is defined on two chains and consists of the electron hopping term, the on-site Coulomb repulsion and a ferromagnetic interaction which describes ferromagnetic coupling between two electrons, one on a bond in a chain and the other on a site in the other chain. It is rigorously shown that the model has fully-polarized ferromagnetic pairing ground states. The higher dimensional version of the model is also presented.

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1. Introduction

Recently, UGe₂ [1], URhGe [2] and UCoGe [3] were discovered to exhibit ferromagnetic superconductivity. Experimental results suggest that the same electrons are contributing to both ferromagnetism and superconductivity, and thus, in the superconducting phase of these materials, the electrons are considered to condensate into a spin-triplet pair state unlike usual low temperature superconductors, in which electrons are forming non-magnetic spin-singlet pairs. Microscopic explanation of this phenomenon is a challenge in condensed matter physics, but the problem is rather subtle and difficult since we have to treat spin-rotation symmetry breaking and electron-pair condensation simultaneously. In fact, the mechanism for ferromagnetism alone in itinerant electron systems has not yet been fully understood, although there have been some rigorous developments [4–9] in the Hubbard model, which is one of the simplest models of itinerant electron systems. As for electron-pair condensation, some models have been shown to have pairing ground states [10–15], but little is known about spin-triplet pairing ground states.

The Coulomb repulsion between electrons combined with the Pauli exclusion principle can generate ferromagnetism in itinerant electron systems [4–9], but it is unclear at the present time that the same can also lead to ferromagnetic superconductivity. Since the electrons form spin-triplet pairs, it is expected that there appears a kind of effective ferromagnetic

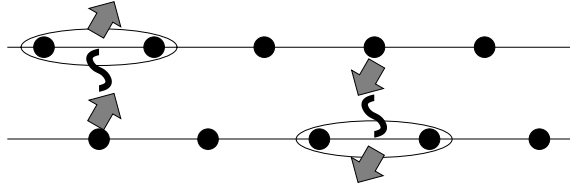


Figure 1. The image of the ferromagnetic interaction considered here. The solid circles represent lattice sites and thick arrows represent electron spins. The electron on a bond in a chain favors to couple ferromagnetically with the one on a site in the other chain.

interaction between electrons in the ferromagnetic superconductors. Thus, a model containing ferromagnetic interactions as well as the Coulomb repulsion between electrons will be suitable for an investigation as a first step toward understanding microscopic mechanisms for ferromagnetic superconductivity.

In this paper, we propose a model which has fully-polarized ferromagnetic pairing ground states. The model consists of the electron hopping term, the on-site Coulomb repulsion and a short-range ferromagnetic interaction term. It is possible to consider the model in any dimension, but for simplicity we mainly concentrate on a one-dimensional version here. In the one-dimensional case, our model is defined on two chains. The electrons can hop along the chain direction, feeling on-site repulsion, and furthermore the two electrons, one on a bond in a chain and the other on a site in the other chain, feel a ferromagnetic interaction (see figure 1). It is shown in our model that, owing to the ferromagnetic interaction introduced here, the electrons form spin-triplet pairs, and then the fully-polarized ferromagnetic pairing state is selected by the on-site Coulomb repulsion as the unique ground state (up to degeneracy due to spin-rotation symmetry). It is also shown that the ground state exhibits off-diagonal long-range order associated with local spin-triplet pairs.

This paper is organized as follows. In the following section, we set up the model and state the main results. In section 3, we prove the main results. In section 4, we make some remarks on our model and ferromagnetic pairing states. We also present models with anisotropic spin interactions and show that the models exhibit other kinds of spin-1 electron-pair condensation. In the final section, we briefly comment on the higher dimensional version of the model.

2. Definition and main results

We start by defining a lattice of our model. Let Λ_1 and Λ_2 be linear chains with L sites. We label sites in Λ_1 and Λ_2 by integers and half integers, respectively, as $\Lambda_1 = \{0, 1, \dots, L-1\}$ and $\Lambda_2 = \{1/2, 3/2, \dots, L-1/2\}$. It is assumed that L is a positive odd integer, and periodic boundary conditions are imposed. Then let $\Lambda = \Lambda_1 \cup \Lambda_2$, on which our Hamiltonian will be defined.

We next introduce fermion operators. As usual, we denote by $c_{x,\sigma}$ and $c_{x,\sigma}^\dagger$ the annihilation and creation operators, respectively, of an electron with spin $\sigma = \uparrow, \downarrow$ at site $x \in \Lambda$. They satisfy the canonical anticommutation relations

$$\{c_{x,\sigma}, c_{y,\tau}\} = \{c_{x,\sigma}^\dagger, c_{y,\tau}^\dagger\} = 0 \quad (2.1)$$

and

$$\{c_{x,\sigma}^\dagger, c_{y,\tau}\} = \delta_{x,y} \delta_{\sigma,\tau} \quad (2.2)$$

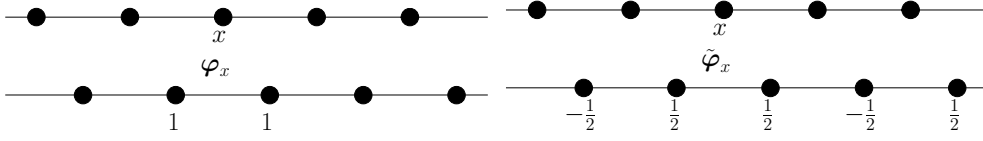


Figure 2. The components of the vectors φ_x and $\tilde{\varphi}_x$.

for any $x, y \in \Lambda$ and $\sigma, \tau = \uparrow, \downarrow$. We denote by N_e the electron number and by Φ_0 a state without electrons. An N_e -electron state can be constructed by operating N_e creation operators $c_{x,\sigma}^\dagger$ on Φ_0 .

We also introduce other fermion operators which play an essential role in our model. For each $x \in \Lambda$, let $\varphi_x = (\varphi_x(y))_{y \in \Lambda}$ be a vector whose components are given by¹

$$\varphi_x(y) = \begin{cases} 1 & \text{if } |y - x| = 1/2, \\ 0 & \text{otherwise,} \end{cases} \quad (2.3)$$

(see figure 2) and define

$$b_{x,\sigma} = \sum_{y \in \Lambda} \varphi_x(y) c_{y,\sigma} = c_{x-\frac{1}{2},\sigma} + c_{x+\frac{1}{2},\sigma}. \quad (2.4)$$

This operator corresponds to a single-electron state localized at a bond of nearest neighbor sites in a chain. Since the set of all vectors φ_x is not orthonormal, the b -operators do not satisfy the canonical anticommutation relations (in fact, it is easy to see that $\{b_{x,\sigma}^\dagger, b_{y,\sigma}\} = 1$ if $|x - y| = 1$), so that we introduce dual operators $\tilde{b}_{x,\sigma}$ as follows. First recall that we adopt the periodic boundary conditions, and thus, for each $x \in \Lambda_1$, sites $y \in \Lambda_2$ can be written as $y = x + 1/2 + n(x, y)$ where $n(x, y)$ is an integer in $0 \leq n(x, y) \leq L - 1$. Then, for $x \in \Lambda_1$, let $\tilde{\varphi}_x = (\tilde{\varphi}_x(y))_{y \in \Lambda}$ be a vector whose components are given by

$$\tilde{\varphi}_x(y) = \begin{cases} 1/2 & \text{if } y \in \Lambda_2 \text{ and } n(x, y) \text{ is even} \\ -1/2 & \text{if } y \in \Lambda_2 \text{ and } n(x, y) \text{ is odd} \\ 0 & \text{if } y \in \Lambda_1, \end{cases} \quad (2.5)$$

(see figure 2) and, for $x \in \Lambda_2$, let $\tilde{\varphi}_x$ be a vector obtained by $\tilde{\varphi}_x(y) = \tilde{\varphi}_{x-1/2}(y - 1/2)$. It is easy to see that the inner product of $\tilde{\varphi}_x$ and φ_y satisfies

$$\langle \tilde{\varphi}_x, \varphi_y \rangle = \sum_{z \in \Lambda} \tilde{\varphi}_x^*(z) \varphi_y(z) = \delta_{x,y}. \quad (2.6)$$

By using $\tilde{\varphi}_x$, we define

$$\tilde{b}_{x,\sigma} = \sum_{y \in \Lambda} \tilde{\varphi}_x(y) c_{y,\sigma}. \quad (2.7)$$

The \tilde{b} -operators are dual to the b -operators in the sense

$$\{\tilde{b}_{x,\sigma}^\dagger, b_{y,\tau}\} = \langle \tilde{\varphi}_x, \varphi_y \rangle \delta_{\sigma,\tau} = \delta_{x,y} \delta_{\sigma,\tau}. \quad (2.8)$$

The anticommutation relation (2.8) implies that the set $\{b_{x,\sigma}^\dagger \Phi_0\}_{x \in \Lambda}$ is linearly independent and so is the set $\{\tilde{b}_{x,\sigma}^\dagger \Phi_0\}_{x \in \Lambda}$. Furthermore, (2.8) implies that the c -operators can be represented

¹ Here $|\cdot|$ means the absolute value. The same symbol $|X|$ is used to denote the number of elements in a set X .

in terms of the \tilde{b} -operators as

$$\begin{aligned} c_{x,\sigma} &= \sum_{y \in \Lambda} \{b_{y,\sigma}^\dagger, c_{x,\sigma}\} \tilde{b}_{y,\sigma} = \sum_{y \in \Lambda} \varphi_y^*(x) \tilde{b}_{y,\sigma} \\ &= \sum_{y \in \Lambda} \varphi_x(y) \tilde{b}_{y,\sigma} = \tilde{b}_{x-\frac{1}{2},\sigma} + \tilde{b}_{x+\frac{1}{2},\sigma}. \end{aligned} \quad (2.9)$$

Let us define the number operators $n_{x,\sigma}$ with $\sigma = \uparrow, \downarrow$ and the spin operators $S_{x,\alpha}$ with $\alpha = 1, 2, 3$ for the c -operators by

$$n_{x,\sigma} = c_{x,\sigma}^\dagger c_{x,\sigma}, \quad (2.10)$$

$$S_{x,1} = \frac{1}{2} (c_{x,\uparrow}^\dagger c_{x,\downarrow} + c_{x,\downarrow}^\dagger c_{x,\uparrow}), \quad (2.11)$$

$$S_{x,2} = \frac{1}{2i} (c_{x,\uparrow}^\dagger c_{x,\downarrow} - c_{x,\downarrow}^\dagger c_{x,\uparrow}), \quad (2.12)$$

$$S_{x,3} = \frac{1}{2} (c_{x,\uparrow}^\dagger c_{x,\uparrow} - c_{x,\downarrow}^\dagger c_{x,\downarrow}). \quad (2.13)$$

We also define

$$n_x = n_{x,\uparrow} + n_{x,\downarrow}. \quad (2.14)$$

The number operators $n_{x,\sigma}^b$ and n_x^b and the spin operators $S_{x,\alpha}^b$ for the b -operators are defined similarly by using $b_{x,\sigma}$ in place of $c_{x,\sigma}$ in (2.10)–(2.14).

By using the operators introduced as above, we define the Hamiltonian H as follows:

$$H_{\text{hop}} = t \sum_{x \in \Lambda} \sum_{\sigma=\uparrow\downarrow} b_{x,\sigma}^\dagger b_{x,\sigma}, \quad (2.15)$$

$$H_{\text{int},J} = -J \sum_{x \in \Lambda} \left(\frac{n_x^b n_x}{4} + \mathbf{S}_x^b \cdot \mathbf{S}_x \right), \quad (2.16)$$

$$H_{\text{int},U} = U \sum_{x \in \Lambda} n_{x,\uparrow} n_{x,\downarrow}, \quad (2.17)$$

$$H = H_{\text{hop}} + H_{\text{int},J} + H_{\text{int},U}, \quad (2.18)$$

where $t, J > 0$ and $U \geq 0$. By using the c -operators, H_{hop} is rewritten as

$$H_{\text{hop}} = t \sum_{x,y \in \Lambda} \sum_{\sigma=\uparrow\downarrow} c_{x,\sigma}^\dagger c_{y,\sigma} + 2t \sum_{x \in \Lambda} n_x, \quad (2.19)$$

$|x-y|=1$

which represents the motion of electrons in the two chains. The Hamiltonian $H_{\text{int},U}$ represents the repulsive interaction between two electrons with the opposite spins at the same site and $H_{\text{hop}} + H_{\text{int},U}$ reduces to the usual Hubbard Hamiltonian. In our model, owing to the Hamiltonian $H_{\text{int},J}$, an electron at a bond in one chain and an electron at a site in the other chain feel the attractive and ferromagnetic interaction. We stress that all the interaction terms considered here are of short-range. It is also noted that the Hamiltonian H conserves the electron number and possesses spin-rotation symmetry. The occurrence of ferromagnetism and electron-pair condensation in our model is thus not trivial at all.

Let $[G_{x,y}]_{x,y \in \Lambda}$ be an antisymmetric matrix whose elements are given by

$$G_{x,y} = \begin{cases} -1/2 & \text{if } x \in \Lambda_1, y \in \Lambda_2 \text{ and } |x-y| = 1/2 \\ 1/2 & \text{if } x \in \Lambda_2, y \in \Lambda_1 \text{ and } |x-y| = 1/2 \\ 0 & \text{otherwise,} \end{cases} \quad (2.20)$$

and define $\tilde{\zeta}_{\sigma\tau}$ and $\zeta_{\sigma\tau}$, corresponding to pair states with spin-1, by

$$\tilde{\zeta}_{\sigma\tau} = \sum_{x,y \in \Lambda} G_{x,y} \tilde{b}_{x,\sigma} \tilde{b}_{y,\tau} \quad (2.21)$$

and

$$\zeta_{\sigma\tau} = \sum_{x,y \in \Lambda} G_{x,y} c_{x,\sigma} c_{y,\tau}, \quad (2.22)$$

respectively.

Our main results are summarized as follows:

Proposition 2.1. *Assume that the electron number N_e is an even integer in $2 \leq N_e \leq |\Lambda|$, and consider the Hamiltonian H with $2t = J$ and $U > 0$. Then the fully-polarized ferromagnetic pairing state,*

$$\Phi_G = (\tilde{\zeta}_{\uparrow\uparrow}^\dagger)^{\frac{N_e}{2}} \Phi_0, \quad (2.23)$$

is the ground state of H . Furthermore, the ground state is unique up to degeneracy due to spin-rotation symmetry.

Proposition 2.2. *Consider the Hamiltonian H with $2t = J$ and $U > 0$, and take a sequence of the ground states Φ_G of H for even N_e and Λ such that the electron filling factor $N_e/(2|\Lambda|)$ converges to ν in $0 < \nu \leq 1/2$. Let $\Delta = \zeta_{\uparrow\uparrow}/L$ and $g_k = 2 \cos(k/2)$. Then we have*

$$\mu(\nu) = \lim_{|\Lambda|, N_e \rightarrow \infty} \frac{\langle \Phi_G, \Delta \Delta^\dagger \Phi_G \rangle}{\langle \Phi_G, \Phi_G \rangle} \geq \frac{1}{2} (1 - 2\nu) I(\nu) \quad (2.24)$$

with

$$I(\nu) = 2 \left(\frac{1}{2\pi} \int_{|k| \leq \pi} \chi[|g_k|^2 \leq \epsilon(\nu)] |g_k|^2 dk \right), \quad (2.25)$$

where $\chi[X]$ equals 1 if X is true and zero otherwise, and $\epsilon(\nu)$ is determined by

$$\nu = \frac{1}{2} \left(\frac{1}{2\pi} \int_{|k| \leq \pi} \chi[|g_k|^2 \leq \epsilon(\nu)] dk \right). \quad (2.26)$$

3. Proof

Proof of proposition 2.1. In the proof we fix the electron number N_e to an even integer in $2 \leq N_e \leq |\Lambda|$.

By some straightforward calculations, we can rewrite $H_{\text{int},J}$ as

$$H_{\text{int},J} = -\frac{J}{2} \sum_{x \in \Lambda} (b_{x,\uparrow}^\dagger b_{x,\uparrow} c_{x,\uparrow}^\dagger c_{x,\uparrow} + b_{x,\downarrow}^\dagger b_{x,\downarrow} c_{x,\downarrow}^\dagger c_{x,\downarrow} + b_{x,\uparrow}^\dagger b_{x,\downarrow} c_{x,\downarrow}^\dagger c_{x,\uparrow} + b_{x,\downarrow}^\dagger b_{x,\uparrow} c_{x,\uparrow}^\dagger c_{x,\downarrow}). \quad (3.1)$$

Noting the assumption $2t = J$ and the anticommutation relations (2.1), (2.2) and $\{b_{x,\sigma}^\dagger, c_{x,\sigma}\} = 0$, we obtain

$$H = \bar{H}_{\text{int},J} + H_{\text{int},U} \quad (3.2)$$

with

$$\bar{H}_{\text{int},J} = \frac{J}{2} \sum_{x \in \Lambda} (b_{x,\uparrow}^\dagger c_{x,\uparrow} + b_{x,\downarrow}^\dagger c_{x,\downarrow})(c_{x,\uparrow}^\dagger b_{x,\uparrow} + c_{x,\downarrow}^\dagger b_{x,\downarrow}). \quad (3.3)$$

Since $\bar{H}_{\text{int},J}$ and $H_{\text{int},U}$ are sums of positive semi-definite operators

$$\frac{J}{2}(b_{x,\uparrow}^\dagger c_{x,\uparrow} + b_{x,\downarrow}^\dagger c_{x,\downarrow})(c_{x,\uparrow}^\dagger b_{x,\uparrow} + c_{x,\downarrow}^\dagger b_{x,\downarrow}) \quad (3.4)$$

and

$$U n_{x,\uparrow} n_{x,\downarrow} = U c_{x,\uparrow}^\dagger c_{x,\downarrow}^\dagger c_{x,\downarrow} c_{x,\uparrow}, \quad (3.5)$$

respectively, the energy eigenvalues of $\bar{H}_{\text{int},J} + H_{\text{int},U}$ are greater than or equal to zero. Therefore, one can conclude that a zero-energy state Φ of both of these Hamiltonians, which satisfies

$$(c_{x,\uparrow}^\dagger b_{x,\uparrow} + c_{x,\downarrow}^\dagger b_{x,\downarrow})\Phi = 0 \quad (3.6)$$

and

$$c_{x,\downarrow} c_{x,\uparrow} \Phi = 0 \quad (3.7)$$

for all $x \in \Lambda$ (if it exists), is a ground state. We shall show that the state Φ_G is indeed a zero-energy state of both $\bar{H}_{\text{int},J}$ and $H_{\text{int},U}$, and thus is a ground state of H .

Since there is no electron with down-spin in Φ_G , the operation of $c_{x,\uparrow}^\dagger b_{x,\uparrow} + c_{x,\downarrow}^\dagger b_{x,\downarrow}$ on Φ_G reduces to $c_{x,\uparrow}^\dagger b_{x,\uparrow} \Phi_G$. Here, by using (2.8) and (2.9), we have

$$c_{x,\uparrow}^\dagger b_{x,\uparrow} \tilde{\zeta}_{\uparrow\uparrow}^\dagger = c_{x,\uparrow}^\dagger \{(-1)^{l+1} c_{x,\uparrow}^\dagger + \tilde{\zeta}_{\uparrow\uparrow}^\dagger b_{x,\uparrow}\} = \tilde{\zeta}_{\uparrow\uparrow}^\dagger c_{x,\uparrow}^\dagger b_{x,\uparrow} \quad (3.8)$$

for $x \in \Lambda_l$ with $l = 1, 2$. Therefore, we have $c_{x,\uparrow}^\dagger b_{x,\uparrow} \Phi_G = 0$, from which $\bar{H}_{\text{int},J} \Phi_G = 0$ follows. Furthermore, noting again that there is no electron with down-spin in Φ_G , one immediately finds $H_{\text{int},U} \Phi_G = 0$. We conclude that Φ_G is a zero-energy state of $\bar{H}_{\text{int},J} + H_{\text{int},U}$.

In what follows, we shall show that any zero-energy state of $\bar{H}_{\text{int},J} + H_{\text{int},U}$ must be expanded in terms of Φ_G and its $SU(2)$ rotations.

Assume that Φ is a zero-energy state of H and thus satisfies (3.6) and (3.7) for all $x \in \Lambda$. We first show that Φ can be written as²

$$\Phi = \sum_{\substack{C \subset \Lambda_1 \\ |C|=N_e/2}} \sum_{\sigma, \tau} f(C; \sigma, \tau) \left(\prod_{x \in C} \tilde{b}_{x, \sigma_x}^\dagger \right) \left(\prod_{x \in C} c_{x, \tau_x}^\dagger \right) \Phi_0, \quad (3.9)$$

with coefficients f , where σ and τ are shorthands of spin configurations $(\sigma_x)_{x \in C}$ and $(\tau_x)_{x \in C}$ with $\sigma_x, \tau_x = \uparrow, \downarrow$, respectively, and the sum $\sum_{\sigma, \tau}$ runs over all possible spin configurations.

To prove the above claim, we begin with preparing basis states for N_e -electron states. Noting the linear independence of \tilde{b} -operators and the relation (2.9), we find that all states of the form

$$\Phi_2(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow) = \left(\prod_{x \in B_\uparrow} \tilde{b}_{x,\uparrow}^\dagger \right) \left(\prod_{x \in B_\downarrow} \tilde{b}_{x,\downarrow}^\dagger \right) \left(\prod_{x \in C_\uparrow} c_{x,\uparrow}^\dagger \right) \left(\prod_{x \in C_\downarrow} c_{x,\downarrow}^\dagger \right) \Phi_0 \quad (3.10)$$

with subsets B_σ, C_σ of Λ_2 such that $\sum_{\sigma=\uparrow,\downarrow} (|B_\sigma| + |C_\sigma|) = N_e$ form complete basis states for the N_e -electron Hilbert space. By using these basis states we expand Φ as

$$\Phi = \sum_{B_\uparrow, B_\downarrow \subset \Lambda_2} \sum_{C_\uparrow, C_\downarrow \subset \Lambda_2} f_2(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow) \Phi_2(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow), \quad (3.11)$$

with coefficients f_2 , where the sum runs over all possible subsets of Λ_2 such that $\sum_{\sigma=\uparrow,\downarrow} (|B_\sigma| + |C_\sigma|) = N_e$. (For notational simplicity, here and in the rest of the proof we do not explicitly write this restriction in the expressions.)

² We assume that the product is ordered according to the labels associated with the sites.

Let us derive the condition on f_2 which follows from (3.6) and (3.7). We choose a site $z \in \Lambda_2$ and $B'_\sigma, C_\sigma \subset \Lambda_2$ with $\sigma = \uparrow, \downarrow$ satisfying $z \notin B'_\sigma$ and $\sum_\sigma (|B'_\sigma| + |C_\sigma|) = N_e - 2$. Then, from (3.6), we have that

$$\left(\prod_{x \in B'_\downarrow} b_{x,\downarrow} \right) \left(\prod_{x \in B'_\uparrow} b_{x,\uparrow} \right) \left(\prod_{x \in C_\downarrow} c_{x,\downarrow} \right) \left(\prod_{x \in C_\uparrow} c_{x,\uparrow} \right) b_{z,-\sigma} c_{z,\sigma} (c_{z,\uparrow}^\dagger b_{z,\uparrow} + c_{z,\downarrow}^\dagger b_{z,\downarrow}) \Phi = 0, \quad (3.12)$$

which leads to $f_2(B'_\uparrow \cup \{z\}, B'_\downarrow \cup \{z\}; C_\uparrow, C_\downarrow) = 0$ for $z \notin C_\sigma$. Therefore, if $z \notin C_\uparrow \cap C_\downarrow$, then $f_2(B'_\uparrow \cup \{z\}, B'_\downarrow \cup \{z\}; C_\uparrow, C_\downarrow) = 0$ or equivalently $f_2(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow) = 0$ for B_\uparrow and B_\downarrow such that $z \in B_\uparrow \cap B_\downarrow$. On the other hand, by examining (3.7) for sites in Λ_2 , we also obtain $f_2(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow) = 0$ for C_\uparrow and C_\downarrow with $C_\uparrow \cap C_\downarrow \neq \emptyset$. As a result, we find that $f_2(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow) = 0$ unless $B_\uparrow \cap B_\downarrow = \emptyset$ as well as $C_\uparrow \cap C_\downarrow = \emptyset$ is satisfied.

We furthermore examine (3.6). Fix C_\uparrow and C_\downarrow with $C_\uparrow \cup C_\downarrow \neq \Lambda_2$ and choose a site $z \in \Lambda_2$ which is not included in $C_\uparrow \cup C_\downarrow$. Using the condition (3.6) for z , we have that

$$\left(\prod_{x \in B'_\downarrow} b_{x,\downarrow} \right) \left(\prod_{x \in B'_\uparrow} b_{x,\uparrow} \right) \left(\prod_{x \in C_\downarrow} c_{x,\downarrow} \right) \left(\prod_{x \in C_\uparrow} c_{x,\uparrow} \right) c_{z,\sigma} (c_{z,\uparrow}^\dagger b_{z,\uparrow} + c_{z,\downarrow}^\dagger b_{z,\downarrow}) \Phi = 0 \quad (3.13)$$

for arbitrary $B'_\uparrow, B'_\downarrow \subset \Lambda_2$ such that $\sum_\sigma (|B'_\sigma| + |C_\sigma|) = N_e - 1$. From this we obtain $f_2(B'_\uparrow \cup \{z\}, B'_\downarrow \cup \{z\}; C_\uparrow, C_\downarrow) = 0$ and $f_2(B'_\uparrow, B'_\downarrow \cup \{z\}; C_\uparrow, C_\downarrow) = 0$ for $z \notin C_\uparrow \cup C_\downarrow$, i.e., that $f_2(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow) = 0$ if there exists a site z such that $z \in B_\uparrow \cup B_\downarrow$ and $z \notin C_\uparrow \cup C_\downarrow$. Therefore, we conclude that $f_2(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow) = 0$ unless all the conditions $C_\uparrow \cap C_\downarrow = \emptyset, B_\uparrow \cap B_\downarrow = \emptyset$ and $(B_\uparrow \cup B_\downarrow) \subset (C_\uparrow \cup C_\downarrow)$ are satisfied.

Taking into account the above result and noting (2.7) and (2.9), we expand Φ as

$$\Phi = \sum_{B_\uparrow, B_\downarrow \subset \Lambda_1} \sum_{C_\uparrow, C_\downarrow \subset \Lambda_1} \chi \left[\sum_\sigma |B_\sigma| \geq \sum_\sigma |C_\sigma| \right] f_1(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow) \Phi_1(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow), \quad (3.14)$$

with coefficients f_1 , where we define Φ_1 as in (3.10), replacing subsets C_σ and B_σ of Λ_2 with those of Λ_1 . Then, repeating the same argument as above, we conclude that $f_1(B_\uparrow, B_\downarrow; C_\uparrow, C_\downarrow) = 0$ unless $C_\uparrow \cap C_\downarrow = \emptyset, B_\uparrow \cap B_\downarrow = \emptyset$ and $(B_\uparrow \cup B_\downarrow) = (C_\uparrow \cup C_\downarrow)$. We thus reach the desired expression (3.9) of a zero-energy state Φ .

We shall next show that the coefficients f in (3.9) satisfy

$$f(C; \sigma, \tau) = f(C'; \sigma', \tau') \quad (3.15)$$

if $\sum_{x \in \Lambda_1} (\sigma_x + \tau_x) = \sum_{x \in \Lambda_1} (\sigma'_x + \tau'_x)$.

For $z \in \Lambda_2$ and $C \subset \Lambda_1$ a straightforward calculation yields

$$\begin{aligned} & c_{z,\downarrow} c_{z,\uparrow} \left(\prod_{x \in C} \tilde{b}_{x,\sigma_x}^\dagger \right) \left(\prod_{x \in C} c_{x,\tau_x}^\dagger \right) \Phi_0 \\ &= \sum_{y, y' \in C} \tilde{\varphi}_y(z) \tilde{\varphi}_{y'}(z) \text{sgn}[y, y'; C] (\chi[\sigma_y = \uparrow, \sigma_{y'} = \downarrow] - \chi[\sigma_y = \downarrow, \sigma_{y'} = \uparrow]) \\ & \quad \times \left(\prod_{x \in C \setminus \{y, y'\}} \tilde{b}_{x,\sigma_x}^\dagger \right) \left(\prod_{x \in C} c_{x,\tau_x}^\dagger \right) \Phi_0. \end{aligned} \quad (3.16)$$

Here $\text{sgn}[\cdot \cdot \cdot]$ is a sign factor arising from exchanges of the fermion operators. By definition, for any y, y' in $C \subset \Lambda_1$, $\tilde{\varphi}_y(z)\tilde{\varphi}_{y'}(z)$ with $z \in \Lambda_2$ is non-vanishing. We thus have from (3.7) and (3.16) that $f(C; \sigma, \tau) = f(C; \sigma[y \leftrightarrow y'], \tau)$, where $\sigma[y \leftrightarrow y']$ is a spin configuration obtained by switching σ_y with $\sigma_{y'}$.

It is also easy to see that

$$\begin{aligned} & (c_{z,\uparrow}^\dagger b_{z,\uparrow} + c_{z,\downarrow}^\dagger b_{z,\downarrow}) \left(\prod_{x \in C} \tilde{b}_{x,\sigma_x}^\dagger \right) \left(\prod_{x \in C} c_{x,\tau_x}^\dagger \right) \Phi_0 \\ &= \text{sgn}[z; C] (\chi[\sigma_z = \uparrow, \tau_z = \downarrow] - \chi[\sigma_z = \downarrow, \tau_z = \uparrow]) \\ & \quad \times c_{z,\uparrow}^\dagger c_{z,\downarrow}^\dagger \left(\prod_{x \in C \setminus \{z\}} \tilde{b}_{x,\sigma_x}^\dagger \right) \left(\prod_{x \in C \setminus \{z\}} c_{x,\tau_x}^\dagger \right) \Phi_0 \end{aligned} \quad (3.17)$$

for $z \in C \subset \Lambda_1$. We thus have from (3.6) and (3.17) that $f(C; \sigma, \tau) = f(C; \sigma[z], \tau[z])$ where $\sigma[z]$ and $\tau[z]$ are obtained by switching σ_z and τ_z in σ and τ respectively.

So far we have shown (3.15) with $C = C'$. We shall complete the proof of the claim by examining (3.6) with $z \in \Lambda_2$.

By using (2.4) and (2.9), (3.6) is rewritten as

$$\sum_{y, y' \in \Lambda_1} \varphi_z(y)\varphi_z(y') (\tilde{b}_{y,\uparrow}^\dagger c_{y',\uparrow} + \tilde{b}_{y,\downarrow}^\dagger c_{y',\downarrow}) \Phi = 0. \quad (3.18)$$

Let y, y' be a pair of sites with $|y - y'| = 1$ in Λ_1 . For this pair, there exists a site z in Λ_2 such that $\varphi_z(y)\varphi_z(y') \neq 0$. Let D be an arbitrary subset of Λ_1 such that $y, y' \notin D$ and $|D| = N_e/2 - 1$. Then, noting (3.18), we obtain from

$$\left(\prod_{x \in D} c_{x,\nu_x} \right) \left(\prod_{x \in D \cup \{y, y'\}} \tilde{b}_{x,\tau_x} \right) (c_{z,\uparrow}^\dagger b_{z,\uparrow} + c_{z,\downarrow}^\dagger b_{z,\downarrow}) \Phi \quad (3.19)$$

$$= \varphi_z(y)\varphi_z(y') \left(\prod_{x \in D} c_{x,\nu_x} \right) \left(\prod_{x \in D \cup \{y, y'\}} \tilde{b}_{x,\tau_x} \right) (\tilde{b}_{y,\tau_y}^\dagger c_{y',\tau_y} + \tilde{b}_{y',\tau_{y'}}^\dagger c_{y,\tau_{y'}}) \Phi = 0 \quad (3.20)$$

that

$$f(D \cup \{y\}; \sigma, \tau) = f(D \cup \{y'\}; \sigma', \tau') \quad (3.21)$$

where $\sigma_y = \tau_{y'}$, $\tau_y = \sigma_{y'}$, $\sigma_{y'} = \tau_y$, $\tau_{y'} = \sigma_y$ and $\sigma_x = \tau'_x = \nu_x$, $\tau_x = \sigma'_x = \tau_x$ for $x \in D$. Let C be an arbitrary subset of Λ_1 with $|C| = N_e/2$ and let $C_{y \rightarrow y'}$ be a subset which is obtained by removing y from and adding y' to C (we assume $y \in C$ and $y' \notin C$). Then the above result implies that (3.15) holds for $C' = C_{y \rightarrow y'}$ if y and y' satisfy $|y - y'| = 1$. It is easy to check that, for arbitrary subsets C and C' of Λ_1 with $|C| = |C'|$, there exist pairs $(\{y_l, y'_l\})_{l=1}^n$ with $|y_l - y'_l| = 1$ such that $C^1 = C_{y_1 \rightarrow y'_1}$, $C^2 = C_{y_2 \rightarrow y'_2}, \dots, C^n = C_{y_n \rightarrow y'_n}^{n-1}$. This completes the proof of the desired relation (3.15).

We write $f(M)$ for $f(C; \sigma, \tau)$ with $\sum_{x \in \Lambda_1} (\sigma_x + \tau_x) = M$. By using (3.15), (3.9) is rewritten as

$$\begin{aligned}
 \Phi &= \sum_{\substack{C \subset \Lambda_1 \\ |C|=N_e/2}} \sum_{M=-N_e/2}^{N_e/2} \sum_{\sigma, \tau} f(M) \chi \left[\sum_{x \in \Lambda_1} (\sigma_x + \tau_x) = M \right] \left(\prod_{x \in C} \tilde{b}_{x, \sigma_x}^\dagger \right) \left(\prod_{x \in C} c_{x, \tau_x}^\dagger \right) \Phi_0 \\
 &= \sum_{\substack{C \subset \Lambda_1 \\ |C|=N_e/2}} \sum_{M=-N_e/2}^{N_e/2} f'(M) (S_{\text{tot}}^-)^{\frac{N_e}{2}-M} \left(\prod_{x \in C} \tilde{b}_{x, \uparrow}^\dagger \right) \left(\prod_{x \in C} c_{x, \uparrow}^\dagger \right) \Phi_0 \\
 &= \sum_{M=-N_e/2}^{N_e/2} f''(M) (S_{\text{tot}}^-)^{\frac{N_e}{2}-M} \Phi_G
 \end{aligned} \tag{3.22}$$

where

$$f'(M) = \frac{(N_e/2 + M)!(N_e/2 - M)!}{N_e!} f(M), \tag{3.23}$$

$$f''(M) = (-1)^{\frac{N_e}{4}(\frac{N_e}{2}-1)} \frac{1}{(N_e/2)!} f'(M), \tag{3.24}$$

and the total spin lowering operator S_{tot}^- is defined as $S_{\text{tot}}^- = \sum_{x \in \Lambda} c_{x, \downarrow}^\dagger c_{x, \uparrow}$. To get the final expression we used

$$\tilde{\zeta}_{\uparrow\uparrow} = \sum_{x \in \Lambda_1, y \in \Lambda_2} G_{x,y} \tilde{b}_{x, \uparrow} \tilde{b}_{y, \uparrow} + \sum_{x \in \Lambda_2, y \in \Lambda_1} G_{x,y} \tilde{b}_{x, \uparrow} \tilde{b}_{y, \uparrow} = \sum_{x \in \Lambda_1} c_{x, \uparrow} \tilde{b}_{x, \uparrow}. \tag{3.25}$$

This completes the proof of proposition 2.1. □

Proof of proposition 2.2. As in the case of $\tilde{\zeta}_{\uparrow\uparrow}$, we have

$$\zeta_{\uparrow\uparrow} = \sum_{x \in \Lambda_1} b_{x, \uparrow} c_{x, \uparrow}. \tag{3.26}$$

The momentum representation of $\zeta_{\uparrow\uparrow}$ is

$$\zeta_{\uparrow\uparrow} = \sum_{k \in \mathcal{K}} g_k \hat{c}_{(2,k), \uparrow} \hat{c}_{(1,-k), \uparrow}, \tag{3.27}$$

where

$$\hat{c}_{(l,k), \sigma} = \frac{1}{\sqrt{L}} \sum_{x \in \Lambda_l} e^{-ikx} c_{x, \sigma}, \tag{3.28}$$

with $l = 1, 2$ and

$$\mathcal{K} = \left\{ k = \frac{2\pi}{L} n \mid -\frac{L-1}{2} \leq n \leq \frac{L-1}{2} \right\}. \tag{3.29}$$

By using (3.25) and (3.26), we have

$$\begin{aligned}
 \Phi_G &\propto \sum_{\substack{C \subset \Lambda_1 \\ |C|=N_e/2}} \left(\prod_{x \in C} \tilde{b}_{x, \uparrow}^\dagger c_{x, \uparrow}^\dagger \right) \Phi_0 \propto \sum_{\substack{C \subset \Lambda_1 \\ |C|=N_e/2}} \left(\prod_{x \in C} \tilde{b}_{x, \uparrow}^\dagger c_{x, \uparrow}^\dagger \right) \left(\prod_{x \in \Lambda_1} b_{x, \uparrow} c_{x, \uparrow} \right) \Phi_F \\
 &\propto \sum_{\substack{C \subset \Lambda_1 \\ |C|=N_e/2}} \left(\prod_{x \in \Lambda_1 \setminus C} b_{x, \uparrow} c_{x, \uparrow} \right) \Phi_F \propto (\zeta_{\uparrow\uparrow})^{\frac{N_h}{2}} \Phi_F
 \end{aligned} \tag{3.30}$$

where $\Phi_F = \left(\prod_{x \in \Lambda} c_{x,\uparrow}^\dagger\right) \Phi_0$ and $N_h = |\Lambda| - N_e$, which is the number of holes counted from half-filling of electrons. We use the above representation of Φ_G with $\zeta_{\uparrow\uparrow}$ in (3.27) to estimate a lower bound on $\mu(\nu)$.

It is easy to check the anticommutation relation $\{\hat{c}_{(l,k),\sigma}^\dagger, \hat{c}_{(l',k'),\sigma'}\} = \delta_{l,l'} \delta_{k,k'} \delta_{\sigma,\sigma'}$, and by using this relation we have

$$\zeta_{\uparrow\uparrow}^\dagger \zeta_{\uparrow\uparrow} = \sum_{k \in \mathcal{K}} |g_k|^2 - \sum_{k \in \mathcal{K}} \sum_{l=1,2} |g_k|^2 \hat{c}_{(l,k),\uparrow} \hat{c}_{(l,k),\uparrow}^\dagger + \zeta_{\uparrow\uparrow} \zeta_{\uparrow\uparrow}^\dagger. \quad (3.31)$$

We furthermore have

$$\hat{c}_{(1,k),\uparrow} \hat{c}_{(1,k),\uparrow}^\dagger \zeta_{\uparrow\uparrow} = g_{-k} \hat{c}_{(2,-k),\uparrow} \hat{c}_{(1,k),\uparrow} + \zeta_{\uparrow\uparrow} \hat{c}_{(1,k),\uparrow} \hat{c}_{(1,k),\uparrow}^\dagger, \quad (3.32)$$

which leads to

$$\hat{c}_{(1,k),\uparrow} \hat{c}_{(1,k),\uparrow}^\dagger (\zeta_{\uparrow\uparrow})^m \Phi_F = m g_{-k} \hat{c}_{(2,-k),\uparrow} \hat{c}_{(1,k),\uparrow} (\zeta_{\uparrow\uparrow})^{m-1} \Phi_F \quad (3.33)$$

for integers m and thus

$$(\zeta_{\uparrow\uparrow})^{\frac{N_h}{2}-m} \hat{c}_{(1,k),\uparrow} \hat{c}_{(1,k),\uparrow}^\dagger (\zeta_{\uparrow\uparrow})^m \Phi_F = \frac{2m}{N_h} \hat{c}_{(1,k),\uparrow} \hat{c}_{(1,k),\uparrow}^\dagger (\zeta_{\uparrow\uparrow})^{\frac{N_h}{2}} \Phi_F. \quad (3.34)$$

The same relation holds for $\hat{c}_{(2,k),\uparrow} \hat{c}_{(2,k),\uparrow}^\dagger$. We thus obtain

$$\begin{aligned} \frac{\langle \Phi_G, \Delta \Delta^\dagger \Phi_G \rangle}{\langle \Phi_G, \Phi_G \rangle} &= \frac{N_h}{2} \frac{1}{L^2} \sum_{k \in \mathcal{K}} |g_k|^2 - \frac{1}{2} \left(\frac{N_h}{2} - 1 \right) \frac{1}{L^2} \sum_{k \in \mathcal{K}} \sum_{l=1,2} |g_k|^2 \frac{\langle \Phi_G, \hat{c}_{(l,k),\uparrow} \hat{c}_{(l,k),\uparrow}^\dagger \Phi_G \rangle}{\langle \Phi_G, \Phi_G \rangle} \\ &\geq \frac{N_h}{4} \frac{1}{L^2} \sum_{k \in \mathcal{K}} \sum_{l=1,2} |g_k|^2 \frac{\langle \Phi_G, \hat{c}_{(l,k),\uparrow} \hat{c}_{(l,k),\uparrow}^\dagger \Phi_G \rangle}{\langle \Phi_G, \Phi_G \rangle} \\ &\geq \frac{|\Lambda| - N_e}{2|\Lambda|} \frac{2}{L} \sum_{m=1}^{N_e/2} |g_{k(m)}|^2, \end{aligned} \quad (3.35)$$

where we arranged the elements in \mathcal{K} as $k(1), k(2), \dots, k(L)$ so that $|g_{k(m)}|^2 \leq |g_{k(m')}|^2$ if $m \leq m'$. Taking the limit $|\Lambda|, N_e \rightarrow \infty$ with $N_e/(2|\Lambda|)$ converging to ν completes the proof. \square

4. Remarks

In this section, we remark some aspects of our model and its ground states.

The state $(S_{\text{tot}}^-)^{\frac{N_e}{2}-M} \Phi_G$ is the representative of the ground state in the subspace where the third component of the total spin is M . If we use the expression (3.30) of Φ_G , this becomes $(\zeta_{\uparrow\uparrow})^{\frac{N_h}{2}} \Phi_M$ where $\Phi_M = (S_{\text{tot}}^-)^{\frac{N_e}{2}-M} \Phi_F$. This implies that the ground state can be regarded as a hole-condensation state in which all holes form the spin-triplet pair in the background of the fully-polarized ferromagnetic state.

By using the so-called d vector $\mathbf{d} = (d_1, d_2, d_3)$, one can obtain a useful representation of a pair operator [16]. In our case, we define

$$\tilde{\zeta}_{\mathbf{d}}^\dagger = \sum_{x,y \in \Lambda_1} \sum_{\sigma,\tau=\uparrow\downarrow} F_{x,y}^{\sigma\tau}(\mathbf{d}) \tilde{b}_{x,\sigma}^\dagger \tilde{b}_{y,\tau}^\dagger \quad (4.1)$$

with

$$\begin{pmatrix} F_{x,y}^{\uparrow\uparrow}(\mathbf{d}) & F_{x,y}^{\uparrow\downarrow}(\mathbf{d}) \\ F_{x,y}^{\downarrow\uparrow}(\mathbf{d}) & F_{x,y}^{\downarrow\downarrow}(\mathbf{d}) \end{pmatrix} = G_{x,y} \begin{pmatrix} -d_1 + id_y & d_3 \\ d_3 & d_1 + id_y \end{pmatrix}. \quad (4.2)$$

The pair operator $\tilde{\zeta}_{\uparrow\uparrow}^\dagger$ corresponds to the case of the complex d vector $\mathbf{d} = (-1/2, -i/2, 0)$, and thus the ground state Φ_G is a nonunitary spin-triplet pairing state similar to the one describing the A_1 -phase of superfluid ^3He in the magnetic field [16]. It is noted that, unlike the case of A_1 -phase realized in the magnetic field, the states

$$\left(\tilde{\zeta}_{\mathbf{d}}^\dagger\right)^{\frac{N_e}{2}} \Phi_0 \quad (4.3)$$

with any \mathbf{d} obtained by rotating $(-1/2, -i/2, 0)$ are the ground states of H , since our Hamiltonian H has spin-rotation symmetry. The direction of the magnetic moment is given by the cross product $i\mathbf{d} \times \mathbf{d}^*$ where $*$ means the complex conjugation.

Let \mathcal{B} be the collection of pairs $\{x, y\}$ of sites such that $x \in \Lambda_1$, $y \in \Lambda_2$ and $|x - y| = 1/2$. Then, we have from (3.26) that

$$\Delta^\dagger = \frac{1}{L} \sum_{\{x,y\} \in \mathcal{B}} c_{x,\uparrow}^\dagger c_{y,\uparrow}^\dagger. \quad (4.4)$$

Therefore, the order parameter $\mu(\nu)$ measures the long-range correlation between local spin-1 electron pairs, and proposition 2.2 implies that there exists this long-range correlation in the ground state of H in the thermodynamic limit.

With respect to the ground state Φ_G expectation values of operators constituted of either annihilation or creation operators, such as Δ and $\hat{c}_{(2,k),\uparrow} \hat{c}_{(1,-k),\uparrow}$, are always zero since there are exactly N_e electrons in Φ_G . In order to obtain a particle number symmetry breaking ground state, which is usually discussed in mean-field approximations, we need to form a linear combination of Φ_G with different electron numbers. Here let us consider the Bardeen–Cooper–Schrieffer-type state

$$\Phi'_G = \prod_{k \in \mathcal{K}} (1 + g_k \hat{c}_{(2,k),\uparrow} \hat{c}_{(1,-k),\uparrow}) \Phi_F. \quad (4.5)$$

It is noted that the projection of Φ'_G onto the Hilbert space with the fixed electron number is proportional to Φ_G and thus Φ'_G attains the ground-state energy of H . Then, for Φ'_G the expectation values of pair annihilation and creation operators are calculated as

$$\frac{\langle \Phi'_G, \hat{c}_{(2,k),\uparrow} \hat{c}_{(1,-k),\uparrow} \Phi'_G \rangle}{\langle \Phi'_G, \Phi'_G \rangle} = \frac{\langle \Phi'_G, \hat{c}_{(1,-k),\uparrow}^\dagger \hat{c}_{(2,k),\uparrow}^\dagger \Phi'_G \rangle}{\langle \Phi'_G, \Phi'_G \rangle} = \frac{gk}{1 + g_k^2}, \quad (4.6)$$

which is finite for $-\pi < k < \pi$.

When we set $U = 0$, the model with $2t = J$ has degenerate ground states and does not exhibit ferromagnetism. In fact, states of the form

$$\left(\tilde{\zeta}_{\downarrow\downarrow}^\dagger\right)^{N_{\downarrow\downarrow}} \left(\tilde{\zeta}_{\uparrow\downarrow}^\dagger\right)^{N_{\uparrow\downarrow}} \left(\tilde{\zeta}_{\uparrow\uparrow}^\dagger\right)^{N_{\uparrow\uparrow}} \Phi_0 \quad (4.7)$$

with non-negative integers $N_{\downarrow\downarrow}$, $N_{\uparrow\downarrow}$ and $N_{\uparrow\uparrow}$ are zero-energy states of $\bar{H}_{\text{int},J}$, since we have

$$\left(c_{x,\uparrow}^\dagger b_{x,\uparrow} + c_{x,\downarrow}^\dagger b_{x,\downarrow}\right) \tilde{\zeta}_{\sigma\tau}^\dagger = \tilde{\zeta}_{\sigma\tau}^\dagger \left(c_{x,\uparrow}^\dagger b_{x,\uparrow} + c_{x,\downarrow}^\dagger b_{x,\downarrow}\right) \quad (4.8)$$

for any $x \in \Lambda$ and $\sigma, \tau = \uparrow, \downarrow$, as in (3.8). The on-site repulsion removes this degeneracy and generates the unique ferromagnetic pairing ground state.

We can extend the present model, which has spin-rotation symmetry, to anisotropic spin-interaction cases. Consider the Hamiltonian

$$H^{\alpha,\beta} = H_{\text{hop}} + H_{\text{int},J}^{\alpha,\beta} + H_{\text{int},U}, \quad (4.9)$$

where $H_{\text{int},J}^{\alpha,\beta}$ is given by

$$H_{\text{int},J}^{\alpha,\beta} = -J \sum_{x \in \Lambda} \left\{ \frac{n_x n_x^b}{4} + \beta S_{x,1}^b S_{x,1} + \beta S_{x,2}^b S_{x,2} + (\alpha - \alpha\beta + \beta) S_{x,3}^b S_{x,3} \right\} \quad (4.10)$$

with $\alpha = \pm 1$ and β in $0 < \beta \leq 1$. It is noted that $H^{\pm 1,1}$ becomes H .

The following results are obtained for the Hamiltonian $H^{\alpha,\beta}$. In the case of $\alpha = 1, 0 < \beta < 1, 2t = J, U > 0$, for even N_e in $2 \leq N_e \leq |\Lambda|$, there exist exactly two ground states of $H^{\alpha,\beta}$, which are given by

$$\Phi_G^{\sigma\sigma} = \left(\tilde{\zeta}_{\sigma\sigma}^\dagger\right)^{\frac{N_e}{2}} \Phi_0 \tag{4.11}$$

with $\sigma = \uparrow, \downarrow$. In this case, therefore, the ground states of $H^{\alpha,\beta}$ exhibit ferromagnetism and condensation of the spin-1 electron pairs whose spins point in the same direction parallel to the third component axis. In the case of $\alpha = -1, 0 < \beta < 1, 2t = J, U = 0$, the ground state of $H^{\alpha,\beta}$ is unique for even N_e in $2 \leq N_e \leq 2|\Lambda|$, and is given by

$$\Phi_G^{\uparrow\downarrow} = \left(\tilde{\zeta}_{\uparrow\downarrow}^\dagger\right)^{\frac{N_e}{2}} \Phi_0. \tag{4.12}$$

In this case the ground state exhibits condensation of the spin-1 electron pairs whose spins point in the plane perpendicular to the third component axis, but does not exhibit ferromagnetism. One can prove these results in the same way as in section 3, by noting the relation

$$H^{\alpha,\beta} = \beta \bar{H}_{\text{int},J} + (1 - \beta) \bar{H}_{\text{int},J}^\alpha + H_{\text{int},U} \tag{4.13}$$

with

$$\bar{H}_{\text{int},J}^\alpha = \frac{J}{2} \sum_{x \in \Lambda} \sum_{\sigma = \uparrow, \downarrow} b_{x,\sigma}^\dagger c_{x,\alpha\sigma} c_{x,\alpha\sigma}^\dagger b_{x,\sigma}. \tag{4.14}$$

In this paper, we treat only the case of $2t = J$ where the exact spin-1 pairing ground states can be constructed. Apart from $2t = J$ we have the following exact results. When we set $J = 0$, the model becomes the decoupled two Hubbard chains, and thus the ground state of H is non-magnetic. In another limit of $U \rightarrow \infty$, where double occupancies at sites are forbidden, it can be proved, by using the Perron–Frobenius theorem³, that the ground state of H for $2t < J, 0 < N_e \leq |\Lambda|$ with open boundary conditions is ferromagnetic. So we expect that the ground-state phase diagram of our model has a rich structure depending on J, U and the electron filling factor ν (and also the anisotropy parameters α and β in the anisotropic cases). It is desirable to describe the phase diagram in detail, but it is beyond our scope at the present time. We leave this problem for future study.

5. Higher dimensional cases

Let L be a positive odd integer and define $\Lambda_1 = [0, L - 1]^d \cap \mathbf{Z}^d$. Let \mathbf{a} be a vector whose all components are $1/2$, and define $\Lambda_2 = \{\mathbf{x} + \mathbf{a} | \mathbf{x} \in \Lambda_1\}$ and $\Lambda = \Lambda_1 \cup \Lambda_2$. We impose periodic boundary conditions in all directions on Λ . For each $\mathbf{x} \in \Lambda$, let $\varphi_{\mathbf{x}} = (\varphi_{\mathbf{x}}(\mathbf{y}))_{\mathbf{y} \in \Lambda}$ be a vector whose components are given by

$$\varphi_{\mathbf{x}}(\mathbf{y}) = \begin{cases} 1 & \text{if } |\mathbf{y} - \mathbf{x}| = |\mathbf{a}| = \sqrt{d/2^d}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Recalling that the periodic boundary conditions are adopted, one notes that for each $\mathbf{x} \in \Lambda_1$, sites $\mathbf{y} \in \Lambda_2$ can be written as $\mathbf{y} = \mathbf{a} + \mathbf{x} + \sum_{l=1}^d n_l(\mathbf{x}, \mathbf{y}) \delta_l$ where δ_l is the unit vector along the l -axis, and $n_l(\mathbf{x}, \mathbf{y})$ with $l = 1, \dots, d$ are integers in $0 \leq n_l(\mathbf{x}, \mathbf{y}) \leq L - 1$. Then, for $\mathbf{x} \in \Lambda_1$, let $\tilde{\varphi}_{\mathbf{x}} = (\tilde{\varphi}_{\mathbf{x}}(\mathbf{y}))_{\mathbf{y} \in \Lambda}$ be a vector whose components are given by

$$\tilde{\varphi}_{\mathbf{x}}(\mathbf{y}) = \begin{cases} 1/2^d & \text{if } \mathbf{y} \in \Lambda_2 \text{ and } \sum_{l=1}^d n_l(\mathbf{x}, \mathbf{y}) \text{ is even} \\ -1/2^d & \text{if } \mathbf{y} \in \Lambda_2 \text{ and } \sum_{l=1}^d n_l(\mathbf{x}, \mathbf{y}) \text{ is odd} \\ 0 & \text{if } \mathbf{y} \in \Lambda_1, \end{cases} \tag{5.2}$$

³ In [6] one can find the detailed explanation for applications of the Perron–Frobenius theorem to ferromagnetism in one dimension.

and for $\mathbf{x} \in \Lambda_2$, let $\tilde{\varphi}_x$ be a vector obtained by $\tilde{\varphi}_x(\mathbf{y}) = \tilde{\varphi}_{x-a}(\mathbf{y} - \mathbf{a})$. Let $[G_{x,y}]_{x,y \in \Lambda}$ be an antisymmetric matrix whose elements are given by

$$G_{x,y} = \begin{cases} -1/2 & \text{if } \mathbf{x} \in \Lambda_1, \mathbf{y} \in \Lambda_2 \text{ and } |\mathbf{x} - \mathbf{y}| = |\mathbf{a}| \\ 1/2 & \text{if } \mathbf{x} \in \Lambda_2, \mathbf{y} \in \Lambda_1 \text{ and } |\mathbf{x} - \mathbf{y}| = |\mathbf{a}| \\ 0 & \text{otherwise.} \end{cases} \quad (5.3)$$

By using φ_x , $\tilde{\varphi}_x$ and $[G_{x,y}]_{x,y \in \Lambda}$ introduced as above, we define b - and \tilde{b} -operators and pair operators $\tilde{\zeta}_{\sigma\tau}$ and $\zeta_{\sigma\tau}$, and then define the Hamiltonian H , as in section 2. For this d -dimensional Hamiltonian we can obtain the same results as in propositions 2.1 and 2.2, where $I(\nu)$ is replaced with

$$I_d(\nu) = 2 \left(\frac{1}{(2\pi)^d} \int_{|k_1| \leq \pi} \cdots \int_{|k_d| \leq \pi} \chi[|g_k|^2 \leq \epsilon(\nu)] |g_k|^2 dk_1 \cdots dk_d \right). \quad (5.4)$$

Here $g_k = 2^d \prod_{l=1}^d \cos(k_l/2)$ and $\epsilon(\nu)$ is determined by

$$\nu = \frac{1}{2} \left(\frac{1}{(2\pi)^d} \int_{|k_1| \leq \pi} \cdots \int_{|k_d| \leq \pi} \chi[|g_k|^2 \leq \epsilon(\nu)] dk_1 \cdots dk_d \right). \quad (5.5)$$

We can also obtain the d -dimensional anisotropic spin-interaction model as in section 4.

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